

1965

## The Generalized Inverse in Linear Programming Basic Theory

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Report Number:  
65-001

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Pyle, L. Duane, "The Generalized Inverse in Linear Programming Basic Theory" (1965). *Department of Computer Science Technical Reports*. Paper 1.  
<https://docs.lib.purdue.edu/cstech/1>

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"The Generalized Inverse in Linear Programming

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by L. Duane Pyle

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### Abstract

Properties of the (Moore-Penrose-Bjerrhammar) generalized inverse  $A^+$  of an arbitrary  $m$  by  $n$  matrix  $A$  are presented and utilized in formulating a linear programming problem, in terms of the eigenvectors of  $I - A^+A$ , which is equivalent to the direct (equalities) form of the linear programming problem. The duality theorem of linear programming is considered from the point of view of this reformulation and a characterization of duality in terms of orthogonality is derived. Other properties of the reformulation are used to characterize edges and extreme points of the convex set of feasible solutions in terms of eigenvectors of certain projection matrices.

"The Generalized Inverse in Linear Programming - Basic Theory"

by L. Duane Pyle

1. Introduction. This is one of a series of papers dealing with different aspects of the same central theme: how the generalized inverse of a matrix and related constructs may be used in connection with linear programming to gain greater understanding of underlying mathematical structure and to provide computational techniques for solution.

Much of the material being presented has previously been rather inaccessible, the principal references being Ph.D. theses [1] and [6] and papers given orally at the 1959 RAND Symposium on Mathematical Programming held in Los Angeles (abstracts were published formally) and at the 1964 International Conference on Mathematical Programming held in London (abstracts were published informally).

In this paper properties of the (Moore-Penrose-Bjerrhammar) generalized inverse  $A^+$  of an arbitrary  $m$  by  $n$  matrix  $A$  are presented and utilized in formulating a linear programming problem, in terms of the eigenvectors of  $I - A^+A$ , which is equivalent to the direct (equalities) form of the linear programming problem. This equivalent problem provides a foundation upon which a number of related studies have been built; some are reported in other papers in this series. The salient features of the derived studies are the following: necessary and sufficient conditions for optimality which lend themselves to constructive realization; a specialization of the Simplex Method for problems having special structures; a characterization of

duality in terms of orthogonality; a characterization of edges and extreme points of the convex set of feasible solutions in terms of the eigenvectors of certain matrices related to  $A$ ; and, combining the last two topics, a formulation of the linear programming problem as a restricted fixed point problem, the restriction being that the fixed point be non-negative.

The last study named gives rise to an iterative computational technique yielding a sequence of vectors converging to a non-negative fixed point. The requirement that the convergence of this sequence be accelerated has resulted in a study of the application of the generalized inverse and the  $\mathcal{E}$ -Algorithm in the construction of intersection projection matrices [6]. The iterative technique described may be used in computing solutions to arbitrary complex nonsingular linear systems, to least squares linear regression problems, as well as to linear programming problems. From the point of view of computation, the indexing and data requirements of the technique are such as to permit utilization of sparsity and/or other structures and sequential vector processing in the sense that at any one time only the matching portions of two vectors are required to be in high speed storage and vectors are treated in a fixed, pre-determined order.

The author is indebted to Dr. A. Charnes for his introduction to the generalized inverse; to Dr. C. F. Kossack, Purdue University and the University of Wisconsin Mathematics Research Center, each having provided material as well as moral support; and to Dr. R. E. Cline, Dr. G. B. Dantzig, Dr. H. Golomb, Dr. T. N. E. Greville, Dr. J. B. Rosen and Dr. P. Wynn.

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## 2. Preliminary Definitions and Theorems

A knowledge of standard notions and manipulations concerning column vectors

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in E^n$$

and matrices having real elements defined on  $E^n$  will be assumed, where  $E^n$  is real, Euclidian  $n$ -dimensional space. Certain results are true for matrices with complex elements but, unless otherwise indicated, it is to be assumed that the elements,  $a_{ij}$ , of a matrix,  $A$ , are real numbers.

The zero vector will be denoted by  $\theta$ , the zero matrix, with dimensions implied by the context, by  $0$ . For emphasis, matrices  $A$  will sometimes be represented either as

$$A = \begin{bmatrix} \leftarrow a^{(1)} \rightarrow \\ \leftarrow \dot{a}^{(1)} \rightarrow \\ \leftarrow \dot{a}^{(m)} \rightarrow \end{bmatrix}$$

where  $a^{(i)}$  is the  $i$ th row vector of  $A$  for  $(i = 1, \dots, m)$ , or as

$$A = \begin{bmatrix} \uparrow c^{(1)} \downarrow & \dots & \uparrow c^{(j)} \downarrow & \dots & \uparrow c^{(n)} \downarrow \end{bmatrix}$$

where  $c^{(j)}$  is the  $j$ th column vector of  $A$  for  $(j=1, \dots, n)$ .

The notation  $(x, c)$ , where  $x$  and  $c$  are vectors, is used to denote the inner product of  $x$  and  $c$ , where

$$(x, c) = \sum_{i=1}^n x_i \bar{c}_i \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$\bar{c}_i$  is the complex conjugate of  $c_i$ ; in most cases under consideration

$c_i$  is real and then  $\bar{c}_i = c_i$ . The length of a vector,  $x$ , is denoted by  $\|x\|$  where

$$\|x\| = \sqrt{(x, x)}.$$

$A^T$  means  $A$  transpose;  $A^*$  means  $A$  complex conjugate transpose.

Although linear programming problems may appear in a variety of forms, it is always possible to obtain an equivalent formulation referred to either as the "Direct Form" of the Linear Programming Problem, (to be abbreviated DLPP) or as the "Equalities Form".

Definition 2.1: The Direct Linear Programming Problem:

Determine the vector,  $x$ , such that the inner product

$$(x, c) = \sum_{j=1}^n c_j x_j \text{ is a maximum where } c \text{ is given and it is required}$$

that  $x \geq 0$  and that  $x$  be a solution of the system of linear equations  $Ax = b$  where  $A$  is a given  $m$  by  $n$  matrix and  $b$  is a given  $m$  by  $1$  vector.

Definition 2.2: The vector  $\bar{x}$  is said to be feasible for the DLPP if both  $\bar{x} \geq 0$  and  $A\bar{x} = b$ .

Definition 2.3: The feasible vector  $\bar{x}$  is said to be optimal for the DLPP if  $(\bar{x}, c) \geq (x, c)$  for all feasible  $x$ .

Associated with a linear programming problem is a problem called its Dual. The Dual of the DLPP is the following:

$$\text{Determine the vector, } w, \text{ such that } (w, b) = \sum_{i=1}^m b_i w_i$$

is a minimum where  $w$  is required to satisfy the system of linear inequalities  $A^T w \leq c$ . Feasible and optimal vectors are defined for the Dual in the same manner as for the DLPP.

These two problems, the DLPP and its Dual, are quite intimately related as is shown by the Duality Theorem stated below. The fundamental nature of the Duality Theorem is illustrated by the fact that necessary and sufficient conditions for the existence of a solution to the DLPP may be shown to follow from its proof [2].

Duality Theorem 2.1: (Gale, Kuhn, Tucker) A feasible  $\bar{x}$  is optimal for the DLPP if and only if there is a  $\bar{w}$  feasible for the Dual with  $(\bar{w}, b) = (\bar{x}, c)$ . A feasible  $\bar{w}$  is optimal for the Dual if and only if there is an  $\bar{x}$  feasible for the DLPP with  $(\bar{x}, c) = (\bar{w}, b)$ .

Existence Theorem 2.2: A necessary and sufficient condition that one of the problems, the DLPP or its Dual, have optimal solutions is that both have feasible solutions.

Since questions related to existence are not to be considered, it will be assumed wherever appropriate throughout this paper that both the DLPP and its Dual have feasible solutions, thus that both have at least one (finite) optimal solution. The following two theorems [5], indicate the direction to be explored:

Theorem 2.3: (Penrose) For every  $m$  by  $n$  matrix,  $A$ , (not necessarily real)  $\exists$  a unique solution,  $X$ , to the following system of matrix equations:

$$(2.1) \quad AXA = A$$

$$(2.2) \quad XAX = X$$

$$(2.3) \quad (AX)^* = AX$$

$$(2.4) \quad (XA)^* = XA$$

The unique matrix,  $X$ , determined by (2.1), (2.2), (2.3), (2.4), is defined to be the generalized inverse of  $A$ , designated as  $A^+$ .



Theorem 2.4: (Penrose) A necessary and sufficient condition for the equation  $Ax = b$  to have a solution is

$$AA^+b = b$$

in which case the general solution is

$$(2.5) \quad x = A^+b + (I - A^+A)y \text{ where } y \text{ is an arbitrary vector.}$$

It is easily verified that

$$(I - A^+A)^2 = I - A^+A \text{ and } (A^+A)^2 = A^+A, \text{ that is,}$$

$$(I - A^+A) \text{ and } A^+A \text{ are idempotent.}$$

$$\text{Also, } (I - A^+A)^T = I - A^+A \text{ and } (A^+A)^T = A^+A, \text{ that is,}$$

$$(I - A^+A) \text{ and } A^+A \text{ are symmetric.}$$

It follows that  $I - A^+A$  and  $A^+A$  are perpendicular projection matrices [4] and that every vector  $z \in E^{(n)}$  can be decomposed as  $z = z^{(1)} + z^{(2)}$  where  $z^{(1)} = (I - A^+A)z$ ,  $z^{(2)} = (A^+A)z$  and the inner product  $(z^{(1)}, z^{(2)}) = 0$ .  $z^{(1)}$  and  $z^{(2)}$  are said to be orthogonal. For simplicity, perpendicular projection matrices will be called projection matrices in the remainder of this paper.

3. The Equivalent Eigenvector Formulation and Duality. Since  $A^+A$  and  $I - A^+A$  are projection matrices, both have only the eigenvalues 0 and 1; since  $A^+A$  is symmetric, the set of eigenvectors of  $A^+A$  contains an orthonormal basis  $e^{(1)}, \dots, e^{(n)}$  of  $E^{(n)}$ . Suppose, for notational convenience, that eigenvectors  $e^{(1)}, \dots, e^{(q)}$ , where  $1 \leq q \leq n-1$ , correspond to the eigenvalue 0 of  $A^+A$ , and that eigenvectors  $e^{(q+1)}, \dots, e^{(n)}$  correspond to the eigenvalue 1. Penrose [5] has shown that  $\text{rank of } A = \text{rank of } A^+A = \text{rank of } A^+ = \text{trace of } A^+A$ . Thus, since the multiplicity,  $(n-q)$ , of  $\lambda = 1$  as an eigenvalue of  $A^+A$  equals the rank of  $A^+A$ ,  $(n-q) = \text{rank of } A^+A = \text{rank of } A$ . It should be observed that

$q = 0$  implies rank of  $A = n$ , i.e.,  $A$  is nonsingular;  $q = n$  implies rank of  $A = 0$ , i.e.,  $A$  is the zero matrix, thus the restriction  $1 \leq q \leq n-1$ .

Theorem 3.1: The general solution (2.5) may be represented as

$$(3.1) \quad x = \sum_{i=q+1}^n \beta_i e^{(i)} + \sum_{i=1}^q \alpha_i e^{(i)}$$

where  $\beta_i = (A^+b, e^{(i)})$ , ( $i = q+1, \dots, n$ )  
and the  $\alpha_i$  are arbitrary, ( $i=1, \dots, q$ ).

Proof: Since the arbitrary vector  $y$  in (2.5) may be expressed as

$$y = \sum_{i=1}^n \alpha_i e^{(i)}$$

where the  $\alpha_i$  are arbitrary, ( $i = 1, \dots, n$ ),

thus

$$(I - A^+A) y = \sum_{i=1}^n \alpha_i (I - A^+A) e^{(i)} = \sum_{i=1}^q \alpha_i e^{(i)}.$$

Since, using (2.2),

$$(I - A^+A) A^+b = (A^+ - A^+A A^+) b = (A^+ - A^+) b = \theta$$

and

$$A^+b = \sum_{i=1}^n (A^+b, e^{(i)}) e^{(i)}$$

thus

$$\begin{aligned} A^+b &= (I - A^+A + A^+A) A^+b \\ &= (I - A^+A) A^+b + A^+A \sum_{i=1}^n (A^+b, e^{(i)}) e^{(i)} \\ &= \theta + \sum_{i=1}^n (A^+b, e^{(i)}) A^+A e^{(i)} \end{aligned}$$

hence

$$A^+b = \sum_{i=q+1}^n (A^+b, e^{(i)}) e^{(i)},$$

For the purpose of notational designation, (3.1) is written below in a slightly different form:

$$(3.2) \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{bmatrix} = \alpha_{q+1} \begin{bmatrix} e_1^{(q+1)} \\ e_2^{(q+1)} \\ \vdots \\ e_k^{(q+1)} \\ \vdots \\ e_n^{(q+1)} \end{bmatrix} + \dots + \alpha_n \begin{bmatrix} e_1^{(n)} \\ e_2^{(n)} \\ \vdots \\ e_k^{(n)} \\ \vdots \\ e_n^{(n)} \end{bmatrix} + \alpha_1 \begin{bmatrix} e_1^{(1)} \\ e_2^{(1)} \\ \vdots \\ e_k^{(1)} \\ \vdots \\ e_n^{(1)} \end{bmatrix} + \dots + \alpha_q \begin{bmatrix} e_1^{(q)} \\ e_2^{(q)} \\ \vdots \\ e_k^{(q)} \\ \vdots \\ e_n^{(q)} \end{bmatrix}$$

Taking inner products on each side of (3.2) with vectors  $e^{(i)}$ , the following relations are obtained:

$$(3.3) \quad \alpha_i = \sum_{j=1}^n e_j^{(i)} x_j \quad (i = 1, \dots, q)$$

$$(3.4) \quad \beta_i = \sum_{j=1}^n e_j^{(i)} x_j \quad (i = q+1, \dots, n)$$

Theorem 3.2: Assuming that  $A A^+ b = b$ , then  $\bar{x}$  is a solution of  $A x = b$  if and only if

$$\beta_i = \sum_{j=1}^n e_j^{(i)} \bar{x}_j; \quad (i = q+1, \dots, n)$$

Proof:  $\bar{x}$  is a solution of  $A x = b$  if and only if  $\bar{x}$  satisfies (3.1) for some combination of  $q$  numbers  $\bar{\alpha}_1, \dots, \bar{\alpha}_q$ .

If  $\bar{x}$  and  $\bar{\alpha}_1, \dots, \bar{\alpha}_q$  satisfy (3.1), then  $\bar{x}$  must satisfy (3.4)

and  $\bar{x}$ , together with  $\bar{\alpha}_1, \dots, \bar{\alpha}_q$ , must satisfy (3.3). Conversely, if  $\bar{x}$  satisfies (3.4), then  $\bar{\alpha}_1, \dots, \bar{\alpha}_q$  may be determined from (3.3) such that  $\bar{x}$  and  $\bar{\alpha}_1, \dots, \bar{\alpha}_q$  satisfy (3.1) and thus  $\bar{x}$  is a solution of  $Ax = b$ .

Geometrically interpreted, the solution of the DLPP is a vector (or vectors)  $x$  which lies in the intersection of the  $q$ -dimensional "flat" (3.1) (if  $q = n-1$ , the term usually used is "hyperplane", rather than flat) and the positive orthant (the  $n$ -dimensional analogue to what in 2-dimensions is the first quadrant) and such that the projection of the vector  $x$  on a fixed vector  $c$  times the length of  $c$ ,  $(x, c)$ , is a maximum.

As a consequence of the representation (3.4) of all solutions to  $Ax = b$ , two questions quite naturally arise: (1) Can a reformulation of the DLPP be obtained, in terms of the eigenvectors of  $I - A^+A$ , equivalent in some sense to the original formulation? (2) Given such a reformulation, does the Duality Theorem have a different interpretation? These questions are answered in Theorem 3.3 and its lemmas.

Definition 3.1: Two linear programming problems, I and II, will be said to be equivalent if the set of feasible solutions of I coincides with the set of feasible solutions of II and if, also, the set of optimal solutions of I coincides with the set of optimal solutions of II.

Consider the linear programming problems A and B defined as follows:

Problem A: Minimize  $(x, -c^q)$  such that  $Ex = \bar{b}$ ,  $x \geq 0$

where  $c^q = (I - A^+A) c$

$$E = \begin{bmatrix} \leftarrow e^{(q+1)T} \rightarrow \\ \vdots \\ \leftarrow e^{(n)T} \rightarrow \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \bar{b} = \begin{bmatrix} (A^+b, e^{(q+1)}) \\ \vdots \\ (A^+b, e^{(n)}) \end{bmatrix} = E A^+b$$

Problem B: Minimize  $(y, A^+b)$  such that  $\bar{E}y = \bar{b}$ ,  $y \geq 0$

where

$$\bar{E} = \begin{bmatrix} \leftarrow e^{(1)T} \rightarrow \\ \vdots \\ \leftarrow e^{(q)T} \rightarrow \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad \bar{b} = \begin{bmatrix} (-c^q, e^{(1)}) \\ \vdots \\ (-c^q, e^{(q)}) \end{bmatrix} = \bar{E} (-c^q)$$

Now, related to linear programming problems A and B are

linear programming problems A' and B' defined as follows:

Problem A': Minimize  $(\alpha, \bar{b})$  such that  $\bar{E}^T \alpha + A^+b \geq 0$

where

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_q \end{bmatrix} \quad \text{is unrestrained, } \bar{E} \text{ and } \bar{b} \text{ defined as in problem B.}$$

Problem B': Minimize  $(\gamma, c)$  such that  $E^T \gamma - c^q \geq 0$

where

$$\gamma = \begin{bmatrix} \gamma_{q+1} \\ \vdots \\ \gamma_n \end{bmatrix} \quad \text{is unrestrained, } E, c \text{ and } c^q \text{ defined as in problem A.}$$

Theorem 3.3: If  $\hat{x}$  and  $\hat{y}$  are feasible for problems A and B, respectively, then  $(\hat{x}, \hat{y}) = 0$  implies  $\hat{x}$  and  $\hat{y}$  are optimal for problems A and B, respectively, and conversely.

Lemma 3.1: Problem A is equivalent to the DLPP: maximize  $(x, c)$  where  $Ax = b$ ,  $x \geq 0$ .

Proof: In the DLPP observe that since the  $e^{(i)}$  ( $i = 1, \dots, n$ ) form an orthonormal basis for  $E^n$ ,  $c$  may be expressed as follows:

$$c = \sum_{i=1}^n (c, e^{(i)}) e^{(i)}.$$

Let 
$$c^q = \sum_{i=1}^q (c, e^{(i)}) e^{(i)} = (I - A^+A) c$$

and 
$$c^{q\perp} = \sum_{i=q+1}^n (c, e^{(i)}) e^{(i)} = A^+A c$$

so that 
$$c = c^q + c^{q\perp} \quad \text{where } (c^q, c^{q\perp}) = 0.$$

Thus 
$$\begin{aligned} (x, c) &= (A^+b + \sum_{i=1}^q a_i e^{(i)}, c^q + c^{q\perp}) \\ &= (A^+b + \sum_{i=1}^q a_i e^{(i)}, c^q) + \\ &\quad (A^+b + \sum_{i=1}^q a_i e^{(i)}, c^{q\perp}) \\ &= (x, c^q) + (A^+b, c^{q\perp}) \\ &= (x, c^q) + (A^+b, A^+Ac) \\ &\quad + (x, c^{q\perp}) + (A^+b, c) \end{aligned}$$

By Theorem 3.2, the solutions of  $Ax = b$  and the solutions of  $\bar{E}x = \bar{b}$  are precisely the same. Thus the set of feasible solutions for problem A and the DLPP problem coincide. Since  $(A^+b, c)$  is a constant, the set of feasible  $x$  such that  $(x, c)$  is a maximum is precisely the same as the set of feasible  $x$  such that  $(x, c^q)$  is a maximum and this proves problem A equivalent to the DLPP.

Observe that although problem A and the DLPP are equivalent, optimal functional values differ by the constant  $(A^+b, c)$  which is readily obtained, given  $A^+$ .

Lemma 3.2: Problem B' is essentially the dual of problem A.

Proof: In problem A replace the requirement, Minimize  $(x, -c^q)$  by the equivalent requirement. Maximize  $(x, c^q)$ .

Lemma 3.3: Problem A' is essentially the dual of problem B.

Proof: Similar to proof of Lemma 3.2.

Lemma 3.4: If  $\bar{x}$  is optimal for problem A, then  $\bar{\alpha} = \bar{E}^T \bar{x}$  is optimal for problem A'. If  $\bar{\alpha}$  is optimal for problem A', then  $\bar{x} = A^+b + \bar{E}^T \bar{\alpha}$  is optimal for problem A.

Proof: If  $\bar{x}$  is a solution of  $E x = b, x \geq \theta$ , then

$$(\bar{x}, e^{(i)}) = (A^+ b, e^{(i)}) \quad (i = q+1, \dots, n)$$

or

$$(\bar{x} - A^+ b, e^{(i)}) = 0$$

Therefore

$$\bar{x} - A^+ b = \sum_{i=1}^q \bar{\alpha}_i e^{(i)} \quad \text{from which, taking inner products,}$$

$$\bar{\alpha}_i = (\bar{x}, e^{(i)}), \quad (i = 1, \dots, q) \text{ is obtained,}$$

or

$$\bar{x} = A^+ b + \bar{E}^T \bar{\alpha} \geq \theta \text{ where } \bar{\alpha} = \bar{E} \bar{x}.$$

Conversely, if  $A^+ b + \bar{E}^T \bar{\alpha} \geq \theta$ , let  $\bar{x} = A^+ b + \bar{E}^T \bar{\alpha}$

$$\text{thus } \bar{x} \geq \theta \text{ and } E \bar{x} = E A^+ b + E \bar{E}^T \bar{\alpha} = b$$

There is thus a 1 to 1 correspondence between vectors which are feasible for problem A and vectors which are feasible for problem A'.

Further, since for feasible corresponding  $x$  and  $\alpha$

$$\begin{aligned} (x, -c^q) &= (A^+ b + \bar{E}^T \alpha, -c^q) = (\bar{E}^T \alpha, -c^q) = (\alpha, \bar{E} (-c^q)) \\ &= (\alpha, \bar{c}), \end{aligned}$$

there is therefore a 1 to 1 correspondence between vectors which are optimal for problem A and vectors which are optimal for problem A'.

It follows that if  $\bar{x}$  is optimal for problem A, then  $\bar{\alpha} = \bar{E} \bar{x}$  is optimal for problem A' and that if  $\bar{\alpha}$  is optimal for problem A', then  $\bar{x} = A^+ b + \bar{E}^T \bar{\alpha}$  is optimal for problem A.



Lemma 3.5: If  $\bar{y}$  is optimal for problem B, then  $\bar{y} = E \bar{y}$  is optimal for problem B'. If  $\bar{y}$  is optimal for problem B', then  $\bar{y} = -c^q + E^T \bar{y}$  is optimal for problem B.

Proof: Follows reasoning similar to that used in proof of Lemma 3.4.

Proof of Theorem 3.3: Let  $\hat{x}$  and  $\hat{y}$  be feasible for problems A and B, respectively.

$$\begin{aligned}
 \text{Then } \hat{x} &= A^+b + \bar{E}^T \hat{\alpha} \geq 0 \quad \text{where } \hat{\alpha} = \bar{E} \hat{x} \\
 \text{and } \hat{y} &= -c^q + E^T \hat{\gamma} \geq 0 \quad \text{where } \hat{\gamma} = E \hat{y} \\
 \text{Thus } (\hat{x}, \hat{y}) &= (A^+b + \bar{E}^T \hat{\alpha}, -c^q + E^T \hat{\gamma}) \\
 &= (A^+b, -c^q) + (A^+b, E^T \hat{\gamma}) + (\bar{E}^T \hat{\alpha}, -c^q) \\
 &\quad + (\bar{E}^T \hat{\alpha}, E^T \hat{\gamma}) \\
 &= (E A^+b, \hat{\gamma}) + (\hat{\alpha}, \bar{E} (-c^q)) \\
 &= (\hat{\gamma}, \theta) + (\hat{\alpha}, \bar{\theta}).
 \end{aligned}$$

By Lemmas 3.4 and 3.5

$$(\hat{x}, -c^q) = (\hat{\alpha}, \bar{\theta})$$

$$\text{and } (\hat{y}, A^+b) = (\hat{\gamma}, \theta).$$

$$\text{Thus } (\hat{x}, \hat{y}) = (\hat{x}, -c^q) + (\hat{y}, A^+b) \geq 0$$

since  $x \geq 0, y \geq 0$

or  $(\hat{x}, c^q) \leq (\hat{y}, A^+b)$  for  $\hat{x}, \hat{y}$  feasible for problems A and B, respectively,

Now, if  $(\hat{x}, \hat{y}) = 0$  where  $\hat{x}$  and  $\hat{y}$  are feasible for problems A and B, respectively, then  $(\hat{x}, c^q) = (\hat{y}, A^+b)$  by previous development.

As before,  $(\hat{\alpha}, \hat{\beta})$

$(\hat{x}, c^q) = \hat{\alpha}$  where  $\hat{\alpha} = \bar{E} \bar{x}$  is feasible for problem A'

and

$(\hat{y}, A^+b) = \hat{\beta}$  where  $\hat{\beta} = E \hat{y}$  is feasible for problem B'.

which, upon substitution in  $(\hat{x}, c^q) = (\hat{y}, A^+b)$ , gives

$$(\hat{x}, -c^q) = -(\hat{y}, \hat{\beta})$$

and

$$(\hat{y}, A^+b) = -(\hat{\alpha}, \hat{\beta}).$$

Therefore, by the Duality Theorem 2.1 and Lemma 3.2 and 3.3

$\hat{x}$  and  $\hat{y}$  are optimal for problems A and B', respectively,

and  $\hat{y}$  and  $\hat{\alpha}$  are optimal for problems B and A', respectively.

Conversely, if  $\hat{x}$  and  $\hat{y}$  are optimal for problems A and B,

respectively, by Lemmas 3.4 and 3.5,  $\hat{\alpha} = \bar{E} \hat{x}$  and  $\hat{\beta} = E \hat{y}$  are

optimal for problems A' and B', respectively, and by state-

ments made in proving Lemmas 3.4 and 3.5

$$(\hat{x}, -c^q) = (\hat{\alpha}, \hat{\beta})$$

$$(\hat{y}, A^+b) = (\hat{\beta}, \hat{\alpha}).$$

But by Lemmas 3.2 and 3.3 and the Duality Theorem 2.1

$$\begin{array}{l} \text{Minimum}_{\text{feasible } x} (x, -c^q) = -(\hat{x}, -c^q) = (\hat{y}, \beta) = \text{Minimum}_{\text{feasible } y} (y, \beta) \end{array}$$

$$\text{and } \begin{array}{l} \text{Minimum}_{\text{feasible } y} (y, A^+b) = -(\hat{y}, A^+b) = (\hat{\alpha}, \bar{\beta}) = \text{Minimum}_{\text{feasible } \alpha} (\alpha, \bar{\beta}) \end{array}$$

therefore

$$-(\hat{x}, -c^q) = (\hat{y}, \beta) = (\hat{y}, A^+b)$$

$$\text{or } (\hat{x}, c^q) = (\hat{y}, A^+b)$$

$$\text{hence } (\hat{x}, \hat{y}) = (\hat{x}, -c^q) + (\hat{y}, A^+b) = 0.$$

4. Other Properties of the Eigenvector Formulation; A Characterization of Edges and Extreme Points. In addition to the Duality Theorem a number of other properties have been studied using the eigenvector formulation. Many of these properties are essential in making certain of the developments given in the other papers in this series.

$$\text{Since } c^q = (I - A^+A)c^q \text{ and}$$

$$A^+b = (A^+A)A^+b$$

$$\text{therefore } \frac{1}{\|c^q\|} c^q = (I - A^+A) \frac{1}{\|c^q\|} c^q$$

$$\text{and } \frac{1}{\|A^+b\|} A^+b = (A^+A) \frac{1}{\|A^+b\|} A^+b .$$

Hence, for any particular problem involving specific vectors  $b$  and  $c$ , the two sets of eigenvectors  $\bar{e}^{(1)}, \dots, \bar{e}^{(q)}$  and  $\bar{e}^{(q+1)}, \dots, \bar{e}^{(n)}$  of  $I - A^+A$  corresponding to  $\lambda=1$  and  $\lambda=0$ , respectively, may be chosen such that

$$\bar{e}^{(1)} = \frac{1}{\|c^q\|} c^q \text{ and } \bar{e}^{(q+1)} = \frac{1}{\|A^+b\|} A^+b .$$

This choice yields the conceptually simple, equivalent formulation of the DLPP:

Determine  $x = \|A^+b\| \bar{e}^{(q+1)} + \sum_{i=1}^q \alpha_i \bar{e}^{(i)} \geq \theta$  such that  $\alpha_i$  is a maximum,  $\alpha_i$  unrestricted.

Geometrically interpreted, an optimal solution is a vector lying in the intersection of the set of solutions to  $Ax=b$  and the positive orthant which goes as far in the direction of the projected gradient  $e^{(1)}$  as it is possible to go. This interpretation suggests necessary and sufficient conditions for optimality as well as a numerical technique for solution (an interior gradient projection method) which form the subject of another paper in this series.

It is possible that  $A^*b=0$  but then  $AA^*b=0$  and, since by Theorem 2.4 a necessary and sufficient condition that  $Ax=b$  is solvable is that  $AA^*b=b$ , this implies that  $b=0$ , in which case the set  $\Lambda$  is a linear manifold passing through the origin and if  $x \in \Lambda$  where  $x \geq 0$ , then  $\alpha x \in \Lambda$  for  $\alpha > 0, \alpha$  arbitrary. Thus  $(x, c)$  is unbounded above and this case has been previously excluded from consideration.

It is possible that  $c^0=0$  but then  $\text{Max}(x, c^0)=0$  and any  $x \geq 0$  such that  $Ax=b$  is optimal. This type of problem is obviously of no practical interest and it will therefore be assumed that  $c^0 \neq 0$ .

Designate the convex set of all feasible solutions to the DLPP as the set  $\Lambda$ ; thus  $\Lambda = \{x | Ax=b, x \geq 0\}$ . Throughout this paper it will be assumed that  $\Lambda$  is bounded and consists of more than one point. It can be shown [2] that the assumption of boundedness implies  $\Lambda$  has a finite number of extreme points,  $E_1, \dots, E_p$ , and

every point  $x \in \Lambda$  may be represented as a convex linear combination of the points  $E_1, \dots, E_p$ ; that is, if  $x \in \Lambda$  then  $\exists \lambda_1, \dots, \lambda_p$  such that

$$x = \sum_{i=1}^p \lambda_i E_i \quad \text{where } \lambda_i \geq 0 \quad \text{and} \quad \sum_{i=1}^p \lambda_i = 1$$

Definition 4.1: The DLPP is incorrectly set if the eigenvectors  $e^{(1)}, \dots, e^{(q)}$  are such that there is at least one row of zeros in the  $n$  by  $q$  matrix whose  $q$  columns are the  $e^{(i)}$  ( $i=1, \dots, q$ ).

A linear programming problem is correctly set if it is not incorrectly set. Incorrectly set DLPP's such that  $\Lambda$  is bounded can be easily constructed as can correctly set DLPP's having unbounded  $\Lambda$ .

Since any extreme point,  $x$ , of  $\Lambda$  can be represented in the form  $x = A^+b + \sum_{i=1}^q \alpha_i e^{(i)}$ , if a problem is incorrectly set, in the " $j$ "th component, say, every extreme point has its " $j$ "th component equal to the " $j$ "th component of  $A^+b$ ,  $(A^+b)_j$ . This means that regardless of the nature of the vector  $c^q$  in  $(x, c^q)$ ,  $x_j = (A^+b)_j$  in an optimal solution. Such behavior indicates that in the initial formulation of the problem, the " $j$ "th variable should not have been included. For this reason it seems appropriate to assume that all problems are correctly set. Any particular problem which is incorrectly set may be recognized either by the form of the  $e^{(i)}$  or by the form of  $I - A^+A$  since, if the problem is incorrectly set, at least one row and the corresponding column of  $I - A^+A$  will be composed entirely of zero elements. Conversely, if in  $I - A^+A$

a row and a corresponding column are composed entirely of zero elements, then each of the eigenvectors for eigenvalue  $\lambda=1$  of  $I-A^*A$  must have zero in the correspondingly placed component, and the problem is incorrectly set.

The properties of  $I-A^*A$  developed previously indicate a natural relationship between the equivalent eigenvector formulation of the DLPP and certain matrix eigenvector problems. This relationship conceivably could be used in numerical solution of linear programming problems since, for the computation of the particular eigenvectors required, the widely used Power Method for solving matrix eigenvalue - eigenvector problems is appropriate. In order to examine this possibility more closely, consider the following:

It is well known [6], [8] that

Theorem 4.1:

$$I-A^*A = \sum_{i=1}^q \frac{1}{e^{(i)}e^{(i)*}} = E^*E$$

and

$$\begin{aligned} A^*A &= \sum_{i=q+1}^n \frac{1}{e^{(i)}e^{(i)*}} = E^*E \\ &= I - \sum_{i=1}^q \frac{1}{e^{(i)}e^{(i)*}} \end{aligned}$$

The edges of  $\Lambda$  joining extreme points are of the form

$$f = \sum_{i=1}^q \gamma_i e^{(i)}, \text{ hence } (I-A^*A)f = f. \text{ That is, the}$$

edges of  $\Lambda$  are eigenvectors of  $(I-A^*A)$  for the eigenvalue  $\lambda=1$ .

Using similar reasoning the following is obtained:

Theorem 4.2: Consider an orthonormal set of eigenvectors of

$$I - A^+A, \text{ say, } \bar{e}^{(i)} \quad (i=1, \dots, n) \quad \text{where} \quad \bar{e}^{(1)} = \frac{1}{\|c^q\|} c^q$$

and

$$\bar{e}^{(q+1)} = \frac{1}{\|A^+b\|} A^+b.$$

Feasible solutions  $x$  and  $y$  to problems  $A$  and  $B$  of Section 3, respectively, are eigenvectors of the projection matrices  $\bar{P}_1$  and  $\bar{P}_2$ , respectively, corresponding to the eigenvalue  $\lambda=1$ , where

$$\bar{P}_1 = \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \bar{e}^{(1)} & \bar{e}^{(2)} & \dots & \bar{e}^{(q)} & \bar{e}^{(q+1)} \\ \uparrow & \uparrow & \uparrow & \uparrow \end{bmatrix} \quad \begin{bmatrix} \leftarrow \bar{e}^{(1)T} \rightarrow \\ \leftarrow \bar{e}^{(2)T} \rightarrow \\ \vdots \\ \leftarrow \bar{e}^{(q)T} \rightarrow \\ \leftarrow \bar{e}^{(q+1)T} \rightarrow \end{bmatrix}$$

and

$$\bar{P}_2 = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \bar{e}^{(q+1)} & \dots & \bar{e}^{(n)} & \bar{e}^{(1)} \\ \uparrow & \uparrow & \uparrow \end{bmatrix} \quad \begin{bmatrix} \leftarrow \bar{e}^{(q+1)T} \rightarrow \\ \vdots \\ \leftarrow \bar{e}^{(n)T} \rightarrow \\ \leftarrow \bar{e}^{(1)T} \rightarrow \end{bmatrix}$$

Proof:

It follows from Theorem 4.1 that  $\bar{P}_1 = I - A^+A + \bar{e}^{(q+1)} \bar{e}^{(q+1)T}$ .

If  $x$  is a solution of  $Ax=b$  then

$$x = A^+b + (I - A^+A)y \quad \text{for some } y.$$



$$\begin{aligned}
\text{Therefore, } \bar{P}_1 x &= (I - A^+ A) A^+ b + (I - A^+ A) y \\
&+ \frac{1}{e^{(q+1)}} \frac{1}{e^{(q+1)T}} [A^+ b + (I - A^+ A) y] \\
&= (I - A^+ A) y + \left( \frac{1}{\|A^+ b\|} A^+ b \right) \left( \frac{1}{\|A^+ b\|} A^+ b \right)^T A^+ b \\
&+ \frac{1}{e^{(q+1)}} \frac{1}{e^{(q+1)T}} \sum_{i=1}^q \alpha_i \frac{1}{e^{(i)}} \\
&= A^+ b + (I - A^+ A) y = x.
\end{aligned}$$

The proof for the second half of the theorem follows in a similar manner.

Corollary 4.1.1: If  $Ax=b$  is solvable and

$$\bar{P}_1 \tilde{x} \equiv (I - A^+ A + \frac{1}{(A^+ b, A^+ b)} A^+ b (A^+ b)^*) \tilde{x} = \tilde{x} \neq \theta$$

where  $(I - A^+ A) \tilde{x} \neq \tilde{x}$  and  $A^+ b \neq \theta$

then  $Ax=b$  where

$$x = \frac{(A^+ b, A^+ b)}{(\tilde{x}, A^+ b)} \tilde{x}$$

Remark: The case  $A^+ b = \theta$  has previously been excluded; if  $(I - A^+ A) \tilde{x} = \tilde{x}$ , then  $x = A^+ b + \mu \tilde{x}$  is a solution of  $Ax=b$  for arbitrary  $\mu$ .

Proof: If  $\bar{P}_1 \tilde{x} = \tilde{x} \neq \theta$  then

$$A^+ A \tilde{x} = \frac{1}{(A^+ b, A^+ b)} A^+ b (A^+ b)^* \tilde{x} = \frac{(\tilde{x}, A^+ b)}{(A^+ b, A^+ b)} A^+ b$$

$$\text{or } AA^+ A \tilde{x} = A \tilde{x} = \frac{(\tilde{x}, A^+ b)}{(A^+ b, A^+ b)} AA^+ b = \frac{(\tilde{x}, A^+ b)}{(A^+ b, A^+ b)} b$$

since  $AA^+ b = b$  if  $Ax=b$  is solvable.

If  $(\tilde{x}, A^+b) = \theta$ ,  $A^+A\tilde{x} = \theta$  or  $(I-A^+A)\tilde{x} = \tilde{x}$ , contrary to assumption. Thus  $A \frac{(A^+b, A^+b)}{(\tilde{x}, A^+b)} \tilde{x} = b$  as required.

Corollary 4.1.2: If  $\bar{E}y = \bar{\beta}$  is solvable ( $\bar{E}$  and  $\bar{\beta}$  defined as in Section 3) and

$$\bar{P}_2 \tilde{y} \equiv (A^+A + \frac{1}{(c^q, c^q)} c^q c^{q*}) \tilde{y} = \tilde{y} \neq \theta$$

where  $A^+A\tilde{y} \neq \tilde{y}$  then  $\bar{E}y = \bar{\beta}$  where  
and  $c^q \neq \theta$   
 $y = \frac{(c^q, c^q)}{(\tilde{y}, -c^q)} \tilde{y}$

Proof: Similar to that of Corollary 4.1.1 after making the substitution  $I-A^+A = \bar{E}^* \bar{E}$  (by Theorem 4.1).

Definition 4.2: Feasible solutions,  $x$  and  $y$ , for problems  $A$  and  $B$  (of Section 3) respectively, will be called basic feasible solutions if there are precisely  $n-q$  and  $q$  non-zero components in  $x$  and  $y$ , respectively, which correspond to linearly independent columns in the matrices  $E$  and  $\bar{E}$ , respectively.

Let  $J$  be an  $n$  by  $n$  matrix whose only non-zero elements are 1's on the diagonal. The zeroes on the diagonal are assumed (for reasons pertinent to the proof of Theorem 4.3) placed in positions corresponding to the positions of  $(q-1)$  zeroes in a basic feasible solution of problem  $A$ . The matrix  $J(I-A^+A)J$  is symmetric and thus all its eigenvalues,  $\lambda_1, \lambda_2, \dots, \lambda_n$ , are real.

Theorem 4.3: Maximum  $\lambda_i = 1$  ( $i=1, \dots, n$ )

Proof: Let  $B = J (I - A^+ A) J$ , then

$$||B|| \leq ||J||^2 ||I - A^+ A||$$

where [3]  $||Q|| = \max_i \sqrt{\mu_i}$  and the  $\mu_i$  ( $i=1, \dots, n$ ) are eigenvalues of  $Q^* Q$ .

Now  $||J|| = ||I - A^+ A|| = 1$ , and

therefore,  $||B|| \leq 1$ . But since maximum  $|\lambda_i|$  of  $B = ||B||$  and since  $B$  obviously has an eigenvalue equal to 1,

$$||B|| = 1 = \text{maximum } \lambda_i \text{ of } J (I - A^+ A) J, \\ (i=1, \dots, n).$$

Using the results of the preceding two theorems, solutions to the DLPP can be calculated using the Power Method for solving  $Bx = x$  by iteration, provided successive  $J$  matrices are properly chosen and an initial, basic feasible solution is known. As will be discussed in more detail in another paper, in this series, certain computational difficulties relative to the placement of zeroes arise when extreme points are encountered which are not basic feasible solutions.

Observe that the same techniques employed in treating problem A as an eigenvector problem can be used in treating problem B.

5. Numerical Example. In order to illustrate portions of the preceding developments, consider the following 2 by 4 Transportation problem:

This particular type DLPP is specified by the following standard rectangular form where the "costs" are internal to the form, the "restrictions" are external:

7	$x_1$	5	$x_2$	8	$x_3$	0	$x_4$	90
2	$x_5$	3	$x_6$	4	$x_7$	0	$x_8$	70
	50		60		40		10	

Such a problem would arise with two "origins" and four "destinations," one of which is "fictitious." The problem is to determine non-negative amounts,  $x_i$ , to place in each cell such that the rows and columns each sum to the indicated restricting amount where it is required that the overall cost be a minimum. The overall cost is obtained by multiplying each cell entry by the associated cell unit cost and summing over all cells. Stated in matrix form, the problem is to minimize  $(x, c)$  such that  $Tx = b$ ,  $x \geq 0$ , where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} \quad c = \begin{bmatrix} 7 \\ 5 \\ 8 \\ 0 \\ 2 \\ 3 \\ 4 \\ 0 \end{bmatrix} \quad b = \begin{bmatrix} 50 \\ 60 \\ 40 \\ 10 \\ 90 \\ 70 \end{bmatrix}$$

and

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

It may be demonstrated that, in this example,

$$T^+ = \frac{1}{48} \begin{bmatrix} 20 & -4 & -4 & -4 & 10 & -2 \\ -4 & 20 & -4 & -4 & 10 & -2 \\ -4 & -4 & 20 & -4 & 10 & -2 \\ -4 & -4 & -4 & 20 & 10 & -2 \\ 20 & -4 & -4 & -4 & -2 & 10 \\ -4 & 20 & -4 & -4 & -2 & 10 \\ -4 & -4 & 20 & -4 & -2 & 10 \\ -4 & -4 & -4 & 20 & -2 & 10 \end{bmatrix}$$

$$T^+T = \frac{1}{8} \begin{bmatrix} 5 & 1 & 1 & 1 & 3 & -1 & -1 & -1 \\ 1 & 5 & 1 & 1 & -1 & 3 & -1 & -1 \\ 1 & 1 & 5 & 1 & -1 & -1 & 3 & -1 \\ 1 & 1 & 1 & 5 & -1 & -1 & -1 & 3 \\ 3 & -1 & -1 & -1 & 5 & 1 & 1 & 1 \\ -1 & 3 & -1 & -1 & 1 & 5 & 1 & 1 \\ -1 & -1 & 3 & -1 & 1 & 1 & 5 & 1 \\ -1 & -1 & -1 & 3 & 1 & 1 & 1 & 5 \end{bmatrix}$$

It is well known that the rank of  $T$  = number of origins plus number of destinations minus one, and thus the rank of  $T = 2 + 4 - 1 = 5$ , and hence  $q = n - \text{rank of } T = 3$ .

Corresponding to  $\lambda=1$  of  $(I-T^+T)$ ,  $e^{(1)}$ ,  $e^{(2)}$  and  $e^{(3)}$  may be chosen as follows:

$$e^{(1)} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad e^{(2)} = \frac{\sqrt{3}}{5} \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \\ -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \quad e^{(3)} = \frac{\sqrt{6}}{12} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \\ -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$$

Corresponding to  $\lambda=0$  of  $(I-T^+T)$ ,  $e^{(4)}, \dots, e^{(8)}$  may be chosen as follows:

$$e^{(4)} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad e^{(5)} = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad e^{(6)} = \frac{\sqrt{6}}{12} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \\ 1 \\ 1 \\ 1 \\ -3 \end{bmatrix} \quad e^{(7)} = \frac{\sqrt{3}}{6} \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} \quad e^{(8)} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

Then

$$T^+b = \frac{5}{2} \begin{bmatrix} 11 \\ 13 \\ 9 \\ 3 \\ 9 \\ 11 \\ 7 \\ 1 \end{bmatrix} \quad \text{and} \quad c^T = \frac{1}{8} \begin{bmatrix} 9 \\ -3 \\ 5 \\ -11 \\ -9 \\ 3 \\ -5 \\ 11 \end{bmatrix}$$

The equivalent eigenvector formulation of the equality constraints analogous to the system  $Ex = \beta$  of Section 3, with common terms cancelled, is given by the following matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -3 & 1 & 1 & 1 & -3 \\ 1 & 1 & -2 & 0 & 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_8 \end{bmatrix} = \begin{bmatrix} 90 \\ 70 \\ 120 \\ 30 \\ -10 \end{bmatrix}$$

In another paper in this series, the general form of  $T^+$ ,  $T^+T$  and the corresponding  $e^{(i)}$  will be exhibited, this being the work of R. Cline [1].

Referring to Section 4, suppose it is known that

$E_{3-6-8} = [20 \ 60 \ 0 \ 10 \ 30 \ 0 \ 40 \ 0]^T$  is an extreme point of the convex set of feasible solutions,  $\Lambda$ . This extreme point is characterized by the coordinate positions which are zero: 3, 6 and 8. Note that the number of zeroes is  $q=3$ , hence  $E_{3-6-8}$  is a non-degenerate extreme point.

From the locations of the zeroes in  $E_{3-6-8}$  it is indicated that there are edges (3-6), (6-8), and (3-8) which are eigenvectors of  $(I-A^+A)$  for  $\lambda=1$ , and by choosing three different  $J$  matrices these edges can be calculated. The terminology "edge" is being used here to designate a vector directed along the line segment connecting two adjacent extreme points.

For example, choosing the  $J$  matrix corresponding to edge (6-8),  
 $B = J (I - T^+ T) J$  is as follows:

$$B = \frac{1}{8} \begin{bmatrix} 3 & -1 & -1 & -1 & -3 & 0 & 1 & 0 \\ -1 & 3 & -1 & -1 & 1 & 0 & 1 & 0 \\ -1 & -1 & 3 & -1 & 1 & 0 & -3 & 0 \\ -1 & -1 & -1 & 3 & 1 & 0 & 1 & 0 \\ -3 & 1 & 1 & 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -3 & 1 & -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is easily verified that  $B$  times edge (6-8) equals  
 edge (6-8) where edge (6-8) =  $[-1 \ 0 \ 1 \ 0 \ 1 \ 0 \ -1 \ 0]^T$ . Edge (6-8)  
 leads to the extreme point  $E_{1-6-8}$ . It can be shown that

$$\text{Edge (3-6)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{leads to} \quad E_{3-4-6} = \begin{bmatrix} 30 \\ 60 \\ 0 \\ 0 \\ 20 \\ 0 \\ 40 \\ 10 \end{bmatrix}$$

and

$$\text{Edge (3-8)} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{leads to} \quad E_{3-5-8} = \begin{bmatrix} 50 \\ 30 \\ 0 \\ 10 \\ 0 \\ 30 \\ 40 \\ 0 \end{bmatrix}$$

where edge (3-6) and edge (3-8) may be obtained in a manner similar  
 to that used in obtaining edge (6-8).



The results of applying the Power Method to  $B$  for a number of iterations are tabulated below. Edge (6-8) is the edge being calculated. The iterative relation being used is  $z^{(k)} = J(I-T^*T) Jz^{(k-1)}$  for  $k=1,2,\dots$  where  $z^{(0)} = [7 \ 5 \ 8 \ 0 \ 2 \ 3 \ 4 \ 0]^T$ .

$$z^{(8)} = \begin{bmatrix} .27121 \\ -.13900 \\ -.22879 \\ -.15855 \\ -.27121 \\ .00000 \\ .22879 \\ .00000 \end{bmatrix} \quad z^{(16)} = \begin{bmatrix} .25909 \\ -.00712 \\ -.24091 \\ -.00719 \\ -.25909 \\ .00000 \\ .24091 \\ .00000 \end{bmatrix} \quad z^{(24)} = \begin{bmatrix} .25098 \\ -.00118 \\ -.24902 \\ -.00118 \\ -.25098 \\ .00000 \\ .24902 \\ .00000 \end{bmatrix}$$

$$z^{(32)} = \begin{bmatrix} .25013 \\ -.00014 \\ -.24987 \\ -.00014 \\ -.25013 \\ .00000 \\ .24987 \\ .00000 \end{bmatrix} \quad z^{(40)} = \begin{bmatrix} .25002 \\ -.00002 \\ -.24998 \\ -.00002 \\ -.25002 \\ .00000 \\ .24998 \\ .00000 \end{bmatrix}$$

Convergence to the vector

$$[.25 \ 0 \ -.25 \ 0 \ -.25 \ 0 \ .25 \ 0]$$

is apparent.

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